Count

16

DMSS1 | 3 | 23-09-16

Today:

- ▷ Exercises listed on homepage for American (pdf) version also
- ▷ My own solutions also on homepage
- ▷ Finish up counting – advanced methods
- ▷ Basic probablity, using counting

Dist obj, dist box	P. 416
Indist obj, dist box	P. 417
Stirling	P. 418
indist, indist	P. 419
Prob complement	P. 435
Prob union	P. 435
Prob cond	P. 442
Prob Indep	P. 443

6.5.24 $\underset{1}{M}\underset{2}{I}\underset{3}{S}\underset{4}{S}\underset{5}{I}\underset{6}{S}\underset{7}{S}\underset{8}{I}\underset{9}{P}\underset{10}{P}\underset{11}{I}$

\# of diff strings, use all letters?

- ▷ Some letters indist
- ▷ Use formula $\frac{n!}{n_1!\,n_2!\cdots n_k!}$ (# of perm of n obj, n_i indist obj of type i)
  Thm 6.5.3

orm
n obj
indist

$\hookrightarrow$ $n = 11$

$n_M = 1$, $n_I = 4$, $n_S = 4$, $n_P = 2$ $\frac{11!}{1!\,4!\,4!\,2!} = 34650$

- ▷ Formula comes from: 11 slots: $\underbrace{\_\ \_\ \cdots\ \_}_{11}$

Choose 1 of 11 for M $\Rightarrow \binom{11}{1}$, slots left: 10

Choose 4 of 10 for I $\Rightarrow \binom{10}{4}$, slots left: 6

S $\Rightarrow \binom{6}{4}$, slots left: 2

P $\Rightarrow \binom{2}{2}$

In total: $\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2}$

Thm 6.5.4:

- ▷ Can be seen as placing dist objs in dist boxes. What are objs and boxes here?

Objs: 11 slots $\underset{1}{\_}\ \underset{2}{\_}\cdots\underset{11}{\_}$

boxes:

$\Rightarrow$ Gives string MISSISSIPPI

√6.5.26 #of diff bit str with 6 × $\boxed{1}$, 8 × $\boxed{0}$
*

▷ Like before: 111111 0000 0000

$n = 14$

$n_1 = 6$, $n_0 = 8$ $\qquad \frac{14!}{6!\,8!} = 3003$

√6.5.30  × 7 to  × 5 from  × 52 (Standard card deck)
* card

▷ Example 6.5.8

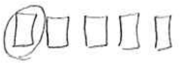
▷ Our case: $\binom{52}{7}\binom{45}{7}\binom{38}{7}\binom{31}{7}\binom{24}{7} = \frac{52!\cdot\cancel{45!}\cdot\cancel{38!}\cdot\cancel{31!}\cdot\cancel{24}}{7!\,\cancel{45!}\cdot7!\,\cancel{38!}\cdot7!\,\cancel{31!}\cdot7!\,\cancel{24!}\cdot7!\cdot17!}$

cards left

$= \frac{52!}{7!\,7!\,7!\,7!\,7!\,17!}$

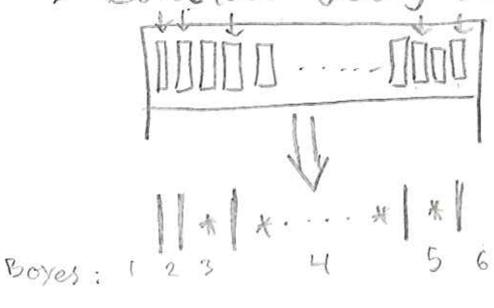
Remaining cards delt to pile on table

√6.5.34   × 12 in row, # of ways to chose 5 books s.t. no two adj books are chosen

▷ If  chosen, then  cant be chosen,  ok

▷ Hint: chosen books: bars
Not chosen books: stars

▷ Consider doing this (Forget requirement of no consecutive books chosen)



Now add requirement, i.e. no box empty (Box = | |)
⟹ Put one in each box by default:
5 bars (chosen books) ⇒ 6 boxes.
Remainding $\overset{12-5=}{7}$ unchosen books must be distributed
So box 2, 3, 4, 5 not empty. 3 * left

$\binom{6+3-1}{3} = 56$

√6.5.38 # of ways: $\underbrace{\circ\circ\circ\circ\circ\circ}_{6,\ \text{dist}} \longrightarrow \underbrace{\sqcup\ \sqcup\ \sqcup\ \sqcup}_{4,\ \text{indist}}$, each box contains ≥ 1 obj
*

▷ Look in heading in 6.5. for dist obj, indist boxes, p. 418

▷ Stirling numbers of second kind (simply called Stirling)

$S(n,j)$: # of ways $\underbrace{\circ\circ\circ\circ\circ\circ}_{n,\ \text{dist}} \longrightarrow \underbrace{\sqcup\sqcup\ldots\sqcup}_{j,\ \text{indist}}$, s.t. no box empty
i.e. each box contains ≥ 1 obj

$$S(n,j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

$\underbrace{\circ\circ\circ\circ\circ\circ\circ}_{k,\ \text{dist}} \curvearrowright \underbrace{\sqcup\sqcup\ldots\sqcup}_{k,\ \text{indist}}$, $\displaystyle\sum_{j=1}^{k} S(n,j)$

▷ What are indist boxes? Put  x 4 in to 3 indist offices.

▷ Our case: $S(6,4) = 65$

√6.5.40 # ways: $\underbrace{\circ\circ\circ\circ\circ}_{5,\ \text{indist}} \longrightarrow \underbrace{\sqcup\ \sqcup\ \sqcup}_{3,\ \text{indist}}$
*

▷ Look in 6.5 for indist obj, indist boxes, p. 419 ⟹ No formula, list all

▷

5	0	0
4	1	0
3	2	0
3	1	1
2	2	1

▷ So result is 5

/7.1.6 p(five card hand contains A♡)

*

▷ What is prob of not happening? $\frac{51}{52}$ for first card, 51 cards left etc.:

$$\frac{\cancel{51}}{52}\cdot\frac{\cancel{50}}{\cancel{51}}\cdot\frac{\cancel{49}}{\cancel{50}}\cdot\frac{\cancel{48}}{\cancel{49}}\cdot\frac{47}{\cancel{48}}=\frac{47}{52}$$

▷ Prob of it happening: $1-\frac{47}{52}=\frac{5}{52}$

▷ $\frac{5}{52}$ makes sense; 5 in hand, 47 on table, $\frac{5}{52}$ chance of being in hand

$\frac{47}{52}$ chance of being on table

▷ Can also be seen as

← choose A♡

$$\frac{\binom{1}{1}\cdot\binom{51}{4}}{\binom{52}{5}}$$

← choose rest

← Total # of hands

▷ Example 7.1.5

√7.1.10 p(five card hand contains exactly one Ace)

(aka p(A♡) or p(A♠) or p(A♢) or p(A♣))

*

▷ Use last formula in prev ex.

4 choices for the ace

$$\frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$$

← choices for rest of hand

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5 kinds in a row

/7.1.14 p(five cards hand is a straight)

▷ Count # straights, then div by $\binom{52}{5}$

▷ Count: Suppose we have hand of straight, order from high to low First card might be $\underset{14}{A}, \underset{13}{K}, \underset{12}{Q}, \underset{11}{J}, 10, 9, 8, \ldots, 5$; The rest of the hand will be completely determined.

Taking suits into account: Each card has 4 poss. suits, 5 cards

$$10\cdot\underbrace{4\cdot4\cdots4}_{5}=10\cdot4^5=10240$$

▷ In total: $\frac{10240}{\binom{52}{5}}$

✓7.1.28 More likely? 🎲 × 2, sum = 8 or 🎲 × 3, sum = 8

* ▷ What does your intuition tell you?

▷ 🎲 × 2: Total outcome $= 6 \cdot 6 = 6^2 = 36$

Sum-8 outcome $= 5$ $\Rightarrow \frac{5}{36} \approx 0{,}13\overline{8}$

Consider $x_1 + x_2 = 8 \Rightarrow x_1 + x_2 = 6 \Rightarrow \binom{2+6-1}{6} - 2 = 5$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

$x_1, x_2 \le 6$

give x_1, x_2 1 by default

-1 for $x_1 = 6$

-1 for $x_2 = 6$

▷ 🎲 × 3: Total outcome $= 6 \cdot 6 \cdot 6 = 6^3 = 216$

Sum-8 outcome $= 21$ $\Rightarrow \frac{21}{216} \approx 0{,}09\overline{72}$

Consider $x_1 + x_2 + x_3 = 8 \Rightarrow x_1 + x_2 + x_3 = 5 \Rightarrow \binom{3+5-1}{5} = 21$

$1 \le x_1 \le 6$

x_2

x_3

give 1 to x_1, x_2, x_3

▷ 🎲 × 2 more likely

7.1.33 [Famous people]

a) ≥ 1 × ⚅ on 🎲 × 4: Prob of not happening: $\frac{5 \cdot 5 \cdot 5 \cdot 5}{6^4} = \frac{5^4}{6^4} = \left(\frac{5}{6}\right)^4 \approx 0{,}48$

roll 4 🎲 w/o ⚅

Prob of happening: $1 - \left(\frac{5}{6}\right)^4 \approx 0{,}517$

b) ≥ 1 × (⚅, ⚅) on (🎲, 🎲) × 24: Same idea, $1 - \frac{35 \cdot 35 \cdots 35}{36^{24}} = 1 - \frac{35^{24}}{36^{24}} \approx 0{,}49$

roll 24 (🎲, 🎲) w/o (⚅, ⚅)

No, prob $\not> \frac{1}{2}$ as it is $\approx 0{,}49$

c) 1 × ⚅ on 🎲 × 4 is more likely

√ 7.2.6 Random permutation of 1, 2, 3, 4

a) 1 precedes 4: By counting: cases

$1\ \_\ \_\ \_ \Rightarrow 3! = 6$

$X\ 1\ \_\ \_ \Rightarrow 2 \cdot 2! = 4$ +

$X\ X\ 1\ 4 \Rightarrow 2! = 2$ +

$= 12$

In total: $4! = 24$

Prob: $\frac{12}{24} = \frac{1}{2}$

b) 4 precedes 1: Must be same as 1 precedes 4 $\Rightarrow \frac{12}{24} = \frac{1}{2}$

▷ Another argument: # 1 prec 4 = # 4 prec 1 (it is symmetrical)
And either 1 prec 4 or 4 prec 1, So $2 \cdot |E| = |S| \Rightarrow p(E) = \frac{|E|}{|S|} = \frac{1}{2}$

c) 4 prec 1, 4 prec 2:

$4\ \_\ \_\ \_ \Rightarrow 3! = 6$

$X\ 4\ \_\ \_ \Rightarrow 2! = 2$ +

$= 8 \Rightarrow \frac{8}{24} = \frac{1}{3}$

d) 4 prec 1, 4 prec 2, 4 prec 3: $4\ \_\ \_\ \_ \Rightarrow 3! = 6 \Rightarrow \frac{6}{24} = \frac{1}{4}$

e) $\underbrace{\text{4 prec 3}}_{E}$, $\underbrace{\text{2 prec 1}}_{F}$: $p(E) = \frac{1}{2}$ (from d) and b)), 2 slots left for 2 and 1. $\frac{1}{2}$ for 2, 1, $\frac{1}{2}$ for 1, 2

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

√ 7.2.10 E, F events, $p(E) = 0{,}8$, $p(F) = 0{,}6$

Show $p(E \cup F) \geq 0{,}8$ and $p(E \cap F) \geq 0{,}4$

▷ $p(E \cup F)$ must be $\geq p(E)$ (read $\cup$ as "or") $\Rightarrow p(E \cup F) \geq 0{,}8$

▷ Use Thm 7.1.2: $p(E \cup F) = p(E) + p(F) - p(E \cap F) = 0{,}8 + 0{,}6 - p(E \cap F)$
$= 1{,}4 - p(E \cap F) \Rightarrow p(E \cap F) \geq 0{,}4$
(To ensure prob not > 1)

7.2.12 Show $p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n)$ ← Boole's ineq

▷ By induction: Basis $n = 2$, Thm 7.1.2 $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
$\leq p(E_1) + p(E_2)$

▷ Assume n, Show n+1: $p(E_1 \cup E_2 \cup \cdots \cup E_n \cup E_{n+1}) \leq p(E_1 \cup \cdots \cup E_n) + p(E_{n+1})$
$\leq p(E_1) + \cdots + p(E_n) + p(E_{n+1})$

induction hypothesis

Base case/ 7.1.2

✓ 7.2.18 Conditional prob 4 x H in 5 coin flips
✗ given first flip come up H

▷ E : 4 x H in 5 coinflips , $P(E) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32}$

F : First flip is H , $P(F) = \frac{1}{2}$

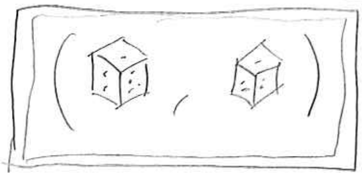
▷ $P(E \cap F) = \frac{1}{2} \cdot P(\text{"} 3 \times H \text{ in last 4 flips"}) = \frac{1}{2} \cdot \frac{\binom{4}{3}}{2^4} = \frac{1}{2} \cdot \frac{4}{16} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

H in first flip

▷ $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

Def 7.2.4

7.2.30



← remote location, ask honest observer
if ≥ 1× ⚅, he answers yes
i.e. we are given that $\underbrace{\geq 1 \times ⚅}_{F}$

a) E : sum = 7
F : ≥ 1 × ⚅

$P(F) = 1 - P(0 \times ⚅ \text{ in 2 rolls}) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$

$P(E \cap F) = \frac{2}{36}$ ← (6, 1) and (1, 6)
← out of all 36 pairs

one ⚅ = 6, so other ⚅ must be 1

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}$

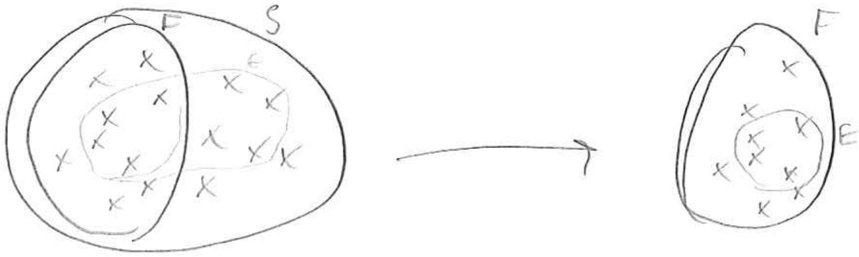
b) E : sum = 7
F : ≥ 1 × ⚄

$P(F) = \frac{11}{36}$ (as before)

$P(E \cap F) = \frac{2}{36}$ ← (5, 2), (2, 5)

$P(E|F) = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}$

Conditional prob talk :



So E gets smaller and F is our new sample space

If we know F happend, one of the picked points must come from F