Announcements

No session next time

Count ~~6~~ ~~7~~ 8

DM551	11	30-11-16

(Mitz) Markov chains p. 153
Recurrence eq p. 498

Today

Markov chains

Recurrence/recursion relations

Revisit $\overset{\text{rand}}{\text{alg}}$ for 2-SAT using Markov chains

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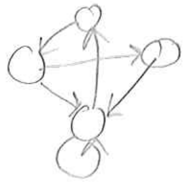
Ex1 on weekly note:

▷ Markov chains (repetition):

Directed graph
(self-loops allowed)



- Each node has name/number and represent a state
- Each edge has a weight w, $0 \leq w \leq 1$ which is the prob of following that edge (all out-edges of node has weight-sum of 1: )

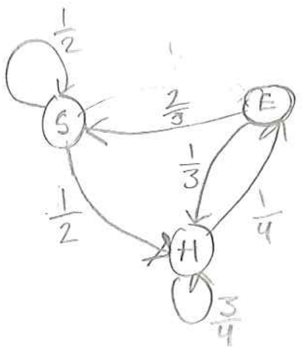
- We start at some node, follow edges with the given prob and continue, potentially for ever.

continued

Ex1 weekly note continued

▷ Example:

Imagine a lion describe by this chain



S: Sleep

H: Hunt

E: Eat

It might stat sleeping, then hunting, eating some prey, hunting some more, eating then sleeping.

We can representing this graph as a matrix

no edge means 0 prob

$$P = \begin{array}{c} \\ S \\ H \\ E \end{array} \begin{array}{ccc} S & H & E \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{array}$$

It is read like: Row = from-state, column = to-state

E.g. $P_{SH} = P_{12} = \frac{1}{2}$ prob of going from S to H.

Also: $S \rightarrow H$ means we transition from S to H.

We can look at different steps in time

random var → X_i = state at step i ; can ask $P(X_i = S)$? etc

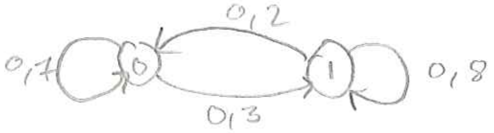
(S/H/E)
(1/2/3)

Now for the exercise

weather describe by chain: $P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{cc} 0 & 1 \\ 0,7 & 0,3 \\ 0,2 & 0,8 \end{array}$

2 states: 0 (sun), 1 (rain)

▷ First, let's draw it: How many nodes?
↳ 2



a) Prob of rainy day followed by sunny?
↳ start at row 1, look at col 0 ⇒ 0,2

Continued

EXI WN continued

b) Formulate in words the event "$X_{62} = 1$"

↳ At day 62 it is raining

c) $p(X_{62} = 1 \mid X_{61} = 0)$?

↳ We know that the day before it was sunny and now we want to know the prob that it is raing.

▷ Just a simple lookup in table or graph.

$\Rightarrow 0{,}3$

Note: It doesn't really matter what the day number is as long as we know the state of yesterday

$p(\square \mid \square)$

defines start state

defines end state

d) Prob that rainy day (start state) is followed by 2 sunny days?

↳ Must! be prob that $1 \rightarrow 0$, then $0 \rightarrow 0$

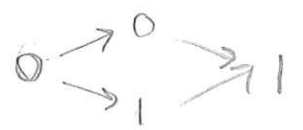
or simply $1 \rightarrow 0 \rightarrow 0$

or $p(x_2 = 0 \mid x_1 = 0) \cdot p(x_1 = 0 \mid x_0 = 1)$ (backwards)

$= 0{,}2 \cdot 0{,}7 = 0{,}14$

e) $p(X_{62} = 1 \mid X_{60} = 0)$? Note: Days are not consecutive!

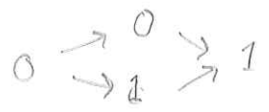
▷ Let's draw "tree" of cases how this can happen



continued

Ex1 WN continued

e)

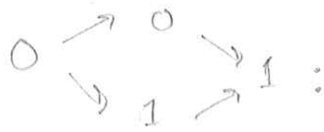


We want to compute prob of 0 → ? → 1 where there are 2 cases for '?'.

As for normal cases we can compute them separately and add them (they have nothing in common so by prob of union (thm 7.1.2) it works)

0 → 0 → 1 : 0,7 · 0,3

0 → 1 → 1 : 0,3 · 0,8



: 0,7 · 0,3 + 0,3 · 0,8 = 0,45

Note that this corresponds to the follow dot-product:

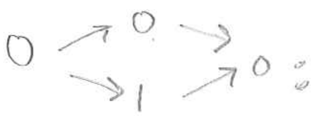
$$\underset{\text{row 1}}{\begin{bmatrix} 0,7 \\ 0,3 \end{bmatrix}} \cdot \underset{\text{col 1}}{\begin{bmatrix} 0,3 \\ 0,8 \end{bmatrix}}$$

$$\begin{pmatrix} 0 \rightarrow 0 \\ 0 \rightarrow 1 \end{pmatrix} \quad \begin{pmatrix} 0 \rightarrow 1 \\ 1 \rightarrow 1 \end{pmatrix}$$

So if we want to answer prob of $\underset{(0/1)}{a} \rightarrow ? \rightarrow \underset{(0/1)}{b}$ we can just do dot prod of the right row and col. It is sort of one step in matrix multiplication! (we will use this fact later)

f) Friday sunny, prob of sunday sunny?

$\text{prob}(X_2 = 0 \mid X_0 = 0)$?



: 0,7 · 0,7 + 0,3 · 0,2 = 0,55

Ex 2 WN

Tourist resort, weather: E_1: $\underset{S}{\text{sun}}$, E_2: $\underset{C}{\text{clouds}}$, E_3: $\underset{R}{\text{rain}}$

$$P = \begin{array}{c} \\ S \\ C \\ R \end{array}\begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0{,}6 & 0{,}2 & 0{,}2 \\ 0{,}3 & 0{,}5 & 0{,}2 \\ 0{,}7 & 0{,}9 & 0{,}3 \end{bmatrix} \end{array}$$

Vacationer wants to visit during December 24-26.
We assume that there is still long time before christmas.

a) Prob of 3 sunny days in a row.

▷ But we don't know our initial state!
We will need something called stationary distribution.
Imagine that we "run" the Markov chain for a very, very long time, we would like to know the prob of being in either of the 3 states.
(The many steps sort of averages out and gives our initial state)
This is why we need to assume that there is a long time before the days of interest (Dec 24-26).

▷ First we need some tools:
Recall that we could answer $0 \to ? \to 1$ with part of matrix mult, we can answer $a \to ? \to b$ for all a and b by multiplying P by itself: $P \cdot P = P^2$.
So P^2 represents $a \to ? \to b$, P^3: $a \to ? \to ? \to b$ etc.
P^n is prob of $\underset{(S/C/R)}{a}$ transitioning to $\underset{(S/C/R)}{b}$ after n days.

We sort of want P^∞, but we do it a bit differently.

We want to find $\pi = [\underset{S}{\pi_1}, \underset{C}{\pi_2}, \underset{R}{\pi_3}]$ ← vector; probs of being in S/C/R.
where

$\pi P = \pi$, In other words:
The matrix P has no effect on π

continued

Ex 2 WN continued

a) We wont to find $\pi P = \pi \Rightarrow [\pi_S, \pi_C, \pi_R] \begin{bmatrix} 0,6 & 0,2 & 0,2 \\ 0,3 & 0,5 & 0,2 \\ 0,7 & 0,0 & 0,3 \end{bmatrix} = [\pi_S, \pi_C, \pi_R]$

We also know $\pi_S + \pi_C + \pi_R = 1$

We have a system of linear equations:

$$\begin{aligned} \pi_S \cdot 0,6 + \pi_C \cdot 0,3 + \pi_R \cdot 0,7 &= \pi_S \\ \pi_S \cdot 0,2 + \pi_C \cdot 0,5 + \pi_R \cdot 0,0 &= \pi_C \\ \pi_S \cdot 0,2 + \pi_C \cdot 0,2 + \pi_R \cdot 0,3 &= \pi_R \\ \pi_S + \pi_C + \pi_R &= 1 \end{aligned}$$

Using Maple this solves to

$$\pi = [\underset{\pi_S}{0,56}, \underset{\pi_C}{0,22}, \underset{\pi_R}{0,22}]$$

I.e. the prob that it is sunny on the initial day is 0,56.

The prob of 3 days sun is

$$\underbrace{0,56}_{\text{start with sun}} \cdot \underbrace{0,6 \cdot 0,6}_{\text{keep sun both days}} = 0,2016$$

b) No rain first 2 days, i.e. start with S or C → S or C

$$\left.\begin{aligned} &\underbrace{0,56}_{\text{init sun}} \cdot \underbrace{(0,6 + 0,2)}_{\text{sun or cloudy day 2}} \\ + &\underbrace{0,22}_{\text{init cloud}} \cdot \underbrace{(0,3 + 0,5)}_{\text{sun or cloudy day 2}} \end{aligned}\right\} = 0,624$$

Jan 15. 1 Recurrence equations (RE)

Some things such as counting and algorithms are often intuitive to describe using recursion. In some courses (eg. DM507) we have seen

$$T(n) = 2\,T(\tfrac{n}{2}) + O(n) \quad \in \Theta(n \log n)$$

In this course we will look at linear RE. Here we only "call" an n at most k smaller i.e. T(n-k), more common written as a_{n-k}

For instance Fibonacci:

$$a_n = a_{n-1} + a_{n-2}$$, here $k = 2$

In general we might have

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_k\, a_{n-k}$$

If no extra term here it is homogenous

But we will focus on REs of degree 2.

We will see how to solve them by example.

a) Solve $a_n = 6 \cdot a_{n-1} - 8 \cdot a_{n-2}$; $\underbrace{a_1 = 4, \quad a_2 = 24}_{\text{base cases/initial conditions}}$

We create <u>characteristic poly</u>:

$$r^2 - c_1 \cdot r - c_2 = 0$$

what is c_1? 6

c_2? -8 $\Rightarrow$ $r^2 - 6 \cdot r + 8 = 0$

Now we find root in ?: $r = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 8}}{2 \cdot 1} = \frac{6 \pm \sqrt{4}}{2}$

$$= \frac{6 \pm 2}{2} = 3 \pm 1 = \begin{cases} 4 \\ 2 \end{cases}$$

Continued

Jan 15.1 continued

a) $r = \begin{cases} 4 \leftarrow r_1 \\ 2 \leftarrow r_2 \end{cases}$

There are multiple cases at this point
In this case char poly has 2 distinct roots

Then the Solution is $a_n = \alpha_1 \cdot r_1^n + \alpha_2 \cdot r_2^n$
where α_1 and α_2 are constants that can be determined by using the base cases.

In our case:

$$a_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot 2^n$$

we can find α_1

n is now 1 $\rightarrow$ $a_1 = 4 = \alpha_1 \cdot 4^1 + \alpha_2 \cdot 2^1 = 4 \cdot \alpha_1 + 2\alpha_2$ (1)

n is now 2 $\rightarrow$ $a_2 = 24 = \alpha_1 \cdot 4^2 + \alpha_2 \cdot 2^2 = 16\alpha_1 + 4\alpha_2$ (2)

$(1): \frac{4}{2} = \frac{4\alpha_1 + 2\alpha_2}{2} \Rightarrow 2 = 2\alpha_1 + \alpha_2 \Rightarrow \alpha_2 = 2 - 2\alpha_1$

$(2)\ \frac{24}{4} = \frac{16\alpha_1 + 4\alpha_2}{4} \Rightarrow 6 = 4\alpha_1 + \alpha_2 \Rightarrow 6 = 4\alpha_1 + 2 - 2\alpha_1$

$\Rightarrow 4 = 2\alpha_1 \Rightarrow \alpha_1 = 2$

$\alpha_2 = 2 - 2 \cdot \alpha_1 = 2 - 2 \cdot 2 = -2$

$a_n = 2 \cdot 4^n - 2 \cdot 2^n$

Jon 15.1 continued

b) Now we have to solve

$$a_n = 6 \cdot a_{n-1} - 8 \cdot a_{n-2} + \overset{F(n)}{3^n} \quad ; \quad a_1 = 9, \; a_2 = 55$$

This is an inhomogenous RE because of 3^n.

The way we solve this is

1. Find solution to homogenous version
2. Find a particular solution to inhom version
3. Add them together.

We already have 1. from a).

How to find 2.? We "guess". (watch video: bit.ly/2fG4hpi)

If F(n) is on a specific form our guess will be

$F(n)$	Guess
c	A
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
c^n	$A c^n$

where A, A_0, A_1, A_2 are just unknown constants

So we guess a particular solution to be $\overset{\text{particular}}{a_n^{(p)}} = A \cdot 3^n$

If we substitute in:

$$A \cdot 3^n = 6 \cdot A 3^{n-1} - 8 \cdot A \cdot 3^{n-2} + 3^n \quad , \text{div by } 3^{n-2}$$

$$\Downarrow$$

$$A \cdot 3^2 = 6 \cdot A \cdot 3^1 - 8 \cdot A \cdot 3^0 + 3^2$$

$$\Downarrow$$

$$A \cdot 9 = A \cdot 18 - A \cdot 8 + 9$$

$$\Downarrow$$

$$A \cdot (9 - 18 + 8) = 9 \Rightarrow A \cdot (-1) = 9 \Rightarrow A = -9$$

$$a_n^{(p)} = -9 \cdot 3^n$$

Solution: $\underbrace{\alpha_1 4^n + \alpha_2 2^n}_{\text{from a)}} \underbrace{- 9 \cdot 3^n}_{a_n^{(p)}}$

continued

Jan 15.1 Continued

b) we now find α_1, α_2 from base cases.

$$a_1 = 9 = \alpha_1 4^1 + \alpha_2 \cdot 2^1 + 9 \cdot 3^1 = \alpha_1 \cdot 4 + \alpha_2 \cdot 2 - 27 \quad (1)$$

$$a_2 = 55 = \alpha_1 4^2 + \alpha_2 \cdot 2^2 - 9 \cdot 3^2 = \alpha_1 \cdot 16 + \alpha_2 \cdot 4 - 81 \quad (2)$$

(1) $9 = \alpha_1 \cdot 4 + \alpha_2 \cdot 2 - 27 \Rightarrow \frac{36}{2} = \frac{\alpha_1 \cdot 4 + \alpha_2 \cdot 2}{2}$

$\Rightarrow 18 = \alpha_1 \cdot 2 + \alpha_2 \Rightarrow \alpha_2 = 18 - \alpha_1 \cdot 2$

(2) $55 = \alpha_1 \cdot 16 + \alpha_2 \cdot 4 - 81 \Rightarrow 55 = \alpha_1 \cdot 16 + (18 - \alpha_1 \cdot 2) \cdot 4 - 81$

$\Rightarrow \alpha_1 = 8$

$\alpha_2 = 18 - \alpha_1 \cdot 2 = 18 - 8 \cdot 2 = 2$

Final solution:

$$a_n = 8 \cdot 4^n + 2 \cdot 2^n - 9 \cdot 3^n$$

Jan 09. 1

Solve $a_n = 2a_{n-1} + 3a_{n-2}$; $a_0 = 1, a_1 = 1$

▷ Order/degree? 2

▷ Hom/inhom? Hom.

▷ Char poly? $r^2 - 2r - 3 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2 = \begin{cases} 3 \\ -1 \end{cases}$$

▷ General solution? (still 2 distinct root)

$$a_n = \alpha_1 3^n + \alpha_2 (-1)^n$$

▷ Find α's:

$$a_0 = 1 = \alpha_1 3^0 + \alpha_2 (-1)^0 = \alpha_1 + \alpha_2 \Rightarrow \alpha_2 = 1 - \alpha_1 \quad (1)$$

$$a_1 = 1 = \alpha_1 3^1 + \alpha_2 (-1)^1 = 3\alpha_1 - \alpha_2$$

$$1 = 3\alpha_1 - (1 - \alpha_1) = 3\alpha_1 - 1 + \alpha_1$$

$$\Downarrow$$

$$2 = 4\alpha_1 \Rightarrow \alpha_1 = \frac{1}{2}$$

$$\alpha_2 = 1 - \alpha_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

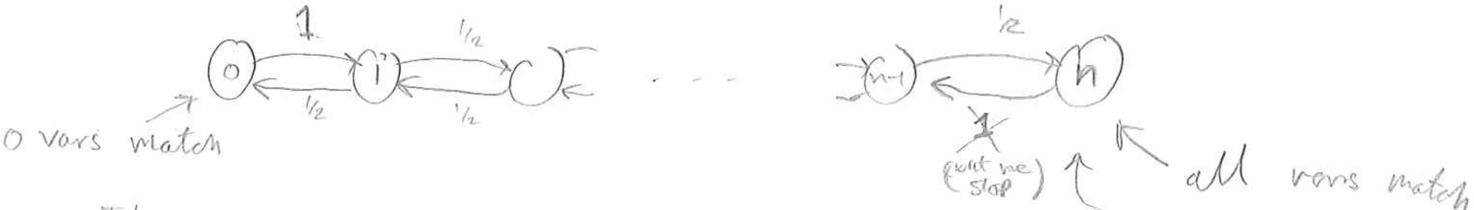
$$a_n = \frac{1}{2} 3^n + \frac{1}{2} (-1)^n$$

Note: If only one root $\overset{r}{\vee}$ to char poly:

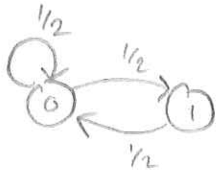
$$a_n = \alpha_1 \cdot r^n + \alpha_2 \cdot n \cdot r^n$$

Mitz 7.6 2-SAT

In Mitz, p. 157, a random alg is used to solve 2-SAT using markov chains.
The idea is that we start with a random assignment and comparing with an optimal solution, they argue that flipping a random literal bring the assignment closer to opt with $\frac{1}{2}$ and away with $\frac{1}{2}$.



They show that after n^2 steps we expect to have reached.

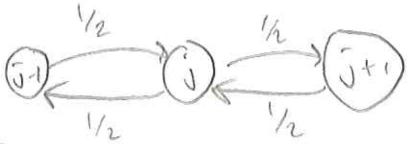
▷ Now the question is:
what if we change  to 
what changes in the analysis?
let's go through it

▷ we still have prob $\geq \frac{1}{2}$ of getting more to the right
Now let h_j = ~~exp~~ # of steps from node j to n.

▷ $h_n = 0$ of course, and
▷ $h_0 = h_1 + 2$ ← was a 1 here before

let Z_j = Random var of # of steps from j to n
So $h_j = E(Z_j)$

Start at node j.



cases: with prob $\frac{1}{2}$: $Z_j = 1 + Z_{j-1}$
(going left)
with prob $\frac{1}{2}$: $Z_j = 1 + Z_{j+1}$

continued

Mitz 7 6 Continued

▷ we get that $E(Z_j) = E\left(\underbrace{\frac{1}{2}(1+Z_{j-1})}_{\text{go left}} + \underbrace{\frac{1}{2}(1+Z_{j+1})}_{\text{go right}}\right)$

Using $h_j = E(Z_j)$ and linearity of exp, we get

$$h_j = \frac{h_{j-1}+1}{2} + \frac{h_{j+1}+1}{2} = \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} + 1 \quad \checkmark \frac{2}{2}$$

So we have

$$h_n = 0$$
$$h_j = \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} + 1$$
$$h_0 = h_1 + 2$$

Next, they show by induction that

$$h_j = h_{j+1} + 2j + 1$$

This doesn't quite work for the case h_0 (base case)

$$h_0 = h_1 + 2\cdot 0 + 2$$

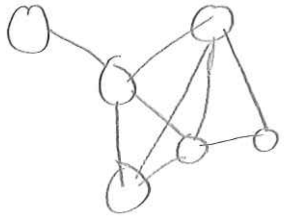
But it is only at h_0 that we have $+2$ instead of $+1$
So ~~we~~ can just do the same and say So just add later

$$h_0 = h_1 + 2\cdot 0 + 1 = h_1 + 1$$
$$= h_2 + 2\cdot 1 + 1 + 1 = h_2 + 3 + 1$$
$$= h_3 + \underset{+1}{2\cdot 2} + 3 + 1 = h_3 + 5 + 3 + 1$$
$$\vdots$$

This is just $\sum_{i=0}^{n-1}(2i+1) = n^2$ $\boxed{+1 \text{ for } h_0}$

So the exp running time is $O_E(n^2)$

Mitz 7.10 Graph coloring



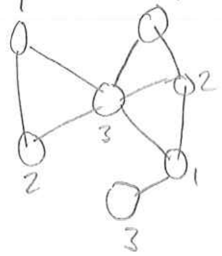
In general:
Answer question: can graph be colored using k colors. (no neighbor nodes must have some color)

In this exercise we let G be a 3-colorable graph.

a) Show ∃ coloring with 2 colors where triangles

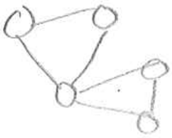
have both colors (ie, no triangle is monochromatic)

↳ suppose we had the 3-coloring

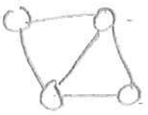


We know all triangles will use all 3 colors and if a triangle [does not touch another triangle then we just change one of the colors to another]

[If some triangles touch they can only touch in one or two nodes



or



]

never mind

If we just change color 3 to 1 or 2 and we are good since all triangles use all three and 3 becomes eg. 1 then all triangles will use 1 and 2.

Continued

Mitz 7.10 continued

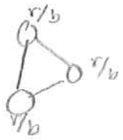
b) we now want such a 2-coloring, but 3-coloring is NP-hard!

Instead we do:

- Start with arbitrary (for instance random) 2-coloring.
- while there are  with same color $(\overset{\text{red}}{1}$ or $\overset{\text{blue}}{2})$
  - flip the color of a random node in 

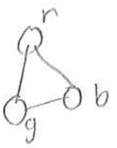
we now want an upper bound on exp recolorings

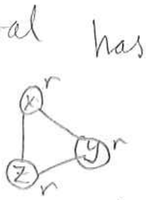
Like with 2-SAT we imagine that we compare our progress with the final result, in this case a valid 2-coloring. Actually we can relax it a bit, and say we compare to a 3-coloring.

When we now take a 

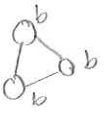
and flip a r or b in node v the following could happen

- v should really be green (3), so it doesn't matter if it is r or b. We are not further or closer to goal
- v should really be r/b, but we just changed it to b/r, so we just got 1 further from goal
- v should really be r/b, and we just changed it to r/b, so we are one step closer to goal.

Each triangle in goal has all three colors 

So when we have 

we got $\frac{1}{3}$ chance of flipping z and not getting closer; $\frac{1}{3}$ chance to flip x and get further away and $\frac{1}{3}$ to flip y and get closer.

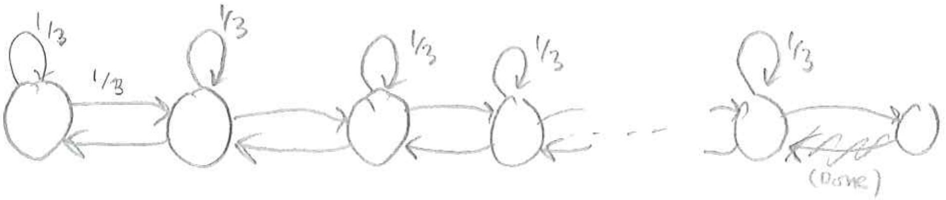
Similar for 

continued

Mitz 7.10 continued

b) So we have $\frac{1}{3}$ for stay, $\frac{1}{3}$ closer by 1, $\frac{1}{3}$ further by 1

We can create Markow chain



Let h_j = exp # of steps from node j to n (similar to before)

The above also gives rise to the following

$$h_j = \underbrace{\frac{h_j+1}{3}}_{\text{stay}} + \underbrace{\frac{h_{j-1}+1}{3}}_{\text{go left}} + \underbrace{\frac{h_{j+1}+1}{3}}_{\text{go right}} = \frac{h_j}{3} + \frac{h_{j-1}}{3} + \frac{h_{j+1}}{3} + \overset{\frac{3}{3}\downarrow}{1}$$

(A bit informal)

The $\frac{h_j}{3}$ we can just imagine splitting up into h_{j-1} and h_{j+1} equally likely, so we can't stay but always go left or right.

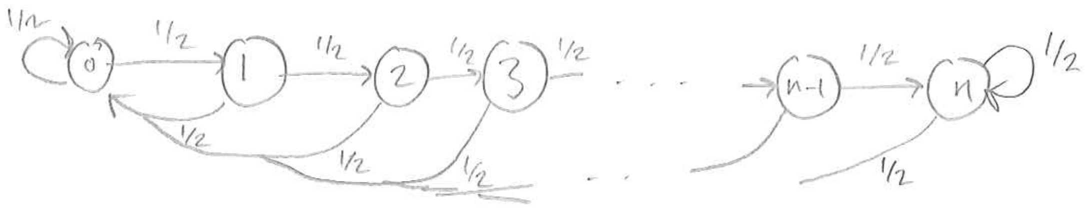
In that case we can just write

$$h_j = \frac{h_{j-1}}{2} + \frac{h_{j+1}}{2} + 1 \implies \underset{E}{O}(n^2)$$

which we just saw previously.

Mitz 7.21

Suppose we have Markov chain like



So we go towards success and at each of the n steps we proceed with $\frac{1}{2}$ prob and are put back to square-one.

▷ Find stationary distribution. (Intuitively we would expect 0 to be more likely)

▷ First write as matrix:

▷ # of coluns? 0 to n-1 (so n+1)
rows? Same, of course.

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & \cdots & & n \\ 0 & 1/2 & 1/2 & 0 & 0 \cdots 0 & & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 \cdots & & 0 \\ 2 & 1/2 & 0 & 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & & & & \ddots & \vdots \\ n-1 & 1/2 & & & & & 1/2 \\ n & 1/2 & & & & & 1/2 \end{array}$$

we now create $\pi P = \pi$

$$\pi_0 \cdot \frac{1}{2} + \pi_1 \frac{1}{2} + \cdots + \pi_n \frac{1}{2} = \pi_0 \quad (1)$$

$$\pi_0 \cdot \frac{1}{2} = \pi_1 \Rightarrow \pi_0 = 2\pi_1$$

$$\pi_1 \cdot \frac{1}{2} = \pi_2$$

$\vdots$

$$\pi_{n-1}\frac{1}{2} + \pi_n \frac{1}{2} = \pi_n \Rightarrow \pi_{n-1}\frac{1}{2} = \pi_n \frac{1}{2}$$

$$\Rightarrow \pi_{n-1} = \pi_n$$

$$\pi_i \frac{1}{2} = \pi_{i+1}$$

(1) $\Rightarrow \pi_0 \frac{1}{2} = \pi_1 \frac{1}{2} + \cdots + \pi_n \frac{1}{2} \Rightarrow \pi_0 = \pi_1 + \cdots + \pi_n$

From this we know that $\pi_0 = \frac{1}{2}$ (as anything would mean $\pi_1 + \pi_2 + \cdots \pi_n \neq \frac{1}{2}$ and the $\sum \pi_i \neq 1$) and each π_i is twice as big as π_{i+1} except π_{n-1} which is $= \pi_n$

Example: $\pi_0 = \frac{1}{2}$, $\pi_1 = \frac{1}{4}$, $\pi_2 = \frac{1}{8}$, $\pi_3 = \frac{1}{8}$