Studie forum, [crossed out] KC 1.2 ons 21

DM551	2	14-09-16

Today:

- Continue counting, more complex, # of ways
- We will talk about rearranging object in slots

  O O O O ⟶ _ _ _ _

  Some times #O = #_

  Some times ∞ #O (rep)

  Sometimes care about order/distinguish O O O O
- We might also relate this to objects into boxes.

  Note: slots ≠ boxes

  Slot: exactly one obj

  box: 0 or more objs

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Permutations	p. 396
Combinations	p. 399
$\binom{n}{r} = \binom{n}{n-r}$	p. 400
Binomial	p. 404
Pascal △	p. 406
Perm + comb table	p. 415
Thm 6.5.1 Perm, Rep	p. 411
Thm 6.5.2 Comb, rep * and |	p. 413

- In the following we will see # ways to choose k obj form n is $\binom{n}{k}$

✓ 6.3.10 # subsets with odd # elements, set of 10 elems

- # subsets (no constraints)? {a, b, c, d, ...}

  $2^n = 2^{10}$

  Bit str 0 1 1 0 ⇒ {b, c, ...}

  1 0 1 0 ⇒ {a, c, ...}

- subsets of odd size

  $2^{n-1} = \frac{2^{10}}{2} = 2^9$

  {a, b}

  1 0 ⇒ {a}

  0 1 ⇒ {b}

# of odd sized subsets containing a = # of even sized subsets not containing a and

# of odd sized subsets not containing a = # even sized subsets containing a

---

✓ 6.3.12 Coin flipped ×10 ⇒ H or T

a) Total outcomes? [H, T, T, T, H, ..] 2^{10}

1 0 0 0 1,

b) Exactly 2 × H [□ □ □ □ □ □ ...] choice 2 of 10 spots for H ⇒ $\binom{10}{2} = 45$

H? H?

c) Atmost 3 × T ⇒ 0×T or 1×T or 2×T or 3×T

$$\underbrace{\binom{10}{0}}_{1} + \underbrace{\binom{10}{1}}_{10} + \underbrace{\binom{10}{2}}_{45} + \underbrace{\binom{10}{3}}_{120} = 176$$

d) #T = #H ⇒ 5×H (and 5×T) ⇒ $\binom{10}{5} = 252$

✓ 6.3.20 * $7 \times ♀$ and $9 \times ♂$ work at faculty

a) # ways, 5 members, $\geq 1 \times ♀$

▷ No restrict $\binom{7+9}{5}$

or $\binom{7}{0}\binom{9}{5} + \underbrace{\binom{7}{1}\binom{9}{4} + \dots + \binom{7}{5}\binom{9}{0}}_{4242}$

▷ # ways to only choose men: $\binom{9}{5} \Rightarrow \binom{7+9}{5} - \binom{9}{5}$ or

b) # ways, 5 members, $\geq 1 \times ♀$ and $1 \times ♂$

▷ # ways only choose women: $\binom{7}{5}$

$\binom{7+9}{5} - \binom{9}{5} - \binom{7}{5}$ or $\binom{7}{1}\binom{9}{4} + \dots + \binom{7}{4}\binom{9}{1}$

check $\binom{7}{1}\binom{12}{4}$

✓ 6.3.24 * Department $10 \times ♂$, $15 \times ♀$

commitee size 6, $\# ♀ > \# ♂$

▷ ~~$0 \times ♀$, $6 \times ♂$~~, ..., ~~$3 \times ♀$, $3 \times ♂$~~, $4 \times ♀$, $2 \times ♂$ (1) ✓, $5 \times ♀$, $1 \times ♂$ (2) ✓, $6 \times ♀$, $0 \times ♂$ (3) ✓

$\binom{15}{4} \cdot \binom{10}{2} + \binom{15}{5}\binom{10}{1} + \binom{15}{6}\binom{10}{0}$

(1) (2) (3)

✓ 6.4.4 * Coefficient of $x^{101} y^{99}$ in $(2x - 3y)^{200} = \overset{(x + y)}{(2x + (-3y))^{200}}$?

▷ Binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\text{coeff}} x^{n-i} y^i = \underbrace{\binom{n}{0}}_{1} x^n + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \underbrace{\binom{n}{n}}_{1} y^n$$

some term

▷ Term $i = 99$: $\binom{200}{99} (2x)^{101} (-3y)^{99} = \underbrace{\binom{200}{99} 2^{101} (-3)^{99}}_{\text{coeff}} x^{101} y^{99}$

✓ 6.4.8 * Pascal's triangle, $\binom{10}{k}$, $0 \leq k \leq 10$:

n : row 10: 1 10 45 120 210 252 210 120 45 10 1

n+1: row 11: 1 11 55 165 330 462 462 330 165 55 11 1

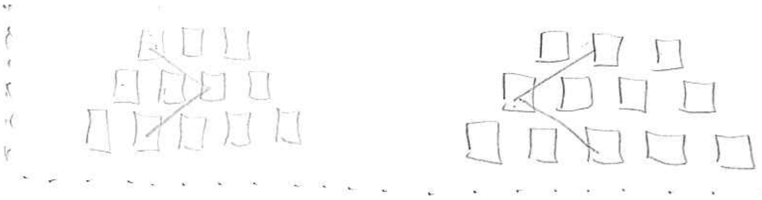
▷ $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

▷ $\binom{n}{k}$ in triangle is row n, column/number k, 0-indexed

6.4.12 Hexagon identity $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$

▷ Rewrite to factorials

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$



$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \frac{(n-1)!}{\underset{1}{(k-1)!}\underbrace{(n-1-(k-1))}_{\substack{n-k\\ 2}}!} \cdot \frac{n!}{\underset{3}{(k+1)!}\underbrace{(n-(k+1))}_{\substack{n-k-1\\ 4}}!} \cdot \frac{(n+1)!}{\underset{5}{k!}\underbrace{(n+1-k)}_{\substack{n-k+1\\ 6}}!}$$

$$\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \frac{(n-1)!}{\underset{5}{k!}\underset{4}{(n-1-k)!}} \cdot \frac{n!}{\underset{1}{(k-1)!}\underbrace{(n-(k-1))}_{\substack{n-k+1\\ 6}}!} \cdot \frac{(n+1)!}{\underset{3}{(k+1)!}\underbrace{(n+1-(k+1))}_{\substack{n-k\\ 2}}!}$$

√6.4.14 $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, $k \le r \le n$

a) Combinatorial proof:

▷ Commitee and subcommitee: First choose r out of n people. Then of these r, choose k.
This is the same as choosing subcommitee of size k, then of the rest choose the normal commitee

b) $$\binom{n}{r}\binom{r}{k} = \frac{n!}{\cancel{r!}(n-r)!} \cdot \frac{\cancel{r!}}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!\cancel{(n-k)!}} \cdot \frac{\cancel{(n-k)!}}{(r-k)!\underbrace{(n-k-(r-k))}_{n-r}!} = \frac{n!}{k!(r-k)!(n-r)!}$$

√6.4.16 p prime, $1 \le k \le p-1$, show $p \mid \binom{p}{k}$

▷ $\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$ $\left(\binom{p}{k} = p \cdot a,\ a \in \mathbb{Z}\right)$

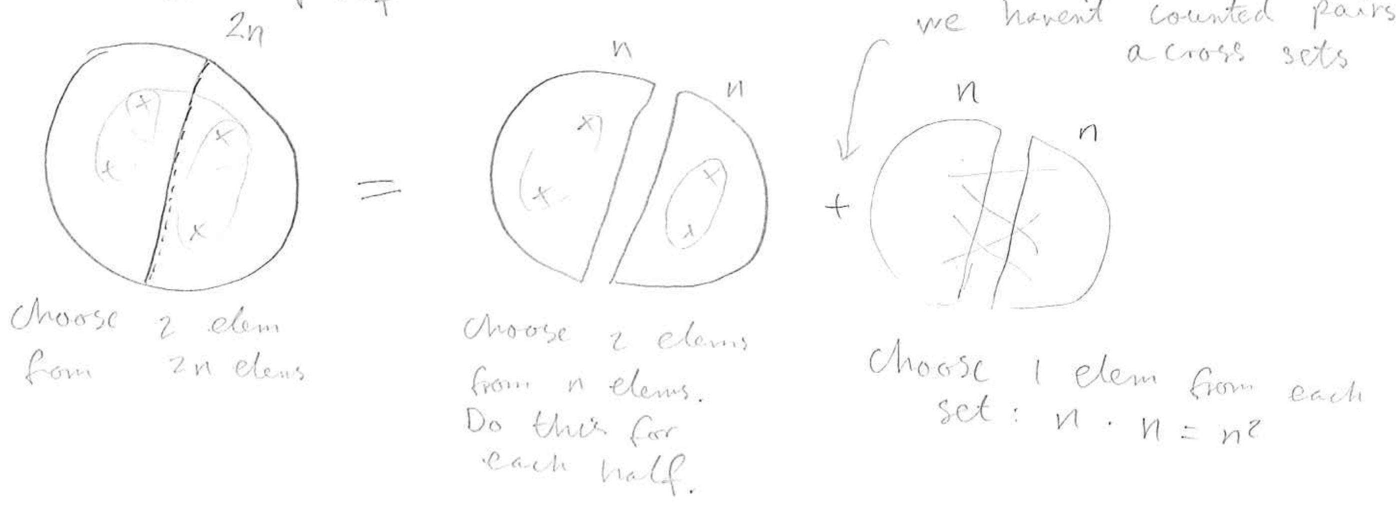
$$= p \underbrace{\frac{(p-1)(p-2)\cdots(p-k+1)}{k!}}_{a}$$

▷ Since none of factors of $k! = k\cdot(k-1)\cdots 1$ can divide p (as p is prime), then it has not been cancelled out and we can split it up in p and a.

▷ a is integer as $\binom{p}{k}$ is integer and $p\cdot(p-1)\cdots(p-k+1) = \binom{p}{k}\cdot k!$

√6.4.20 Show $\binom{2n}{2} = 2\binom{n}{2} + n^2$

a) combinatorial proof



b)

$$\binom{2n}{2} = \frac{(2n)!}{\underbrace{2!}_{2}(2n-2)!} = \frac{1}{2}\,\frac{2n\cdot(2n-1)\cdot\cancel{(2n-2)\cdots n\cdot(n-1)\cdots 1}}{\cancel{(2n-2)\cdots n\cdot(n-1)\cdots 1}} = n\cdot(2n-1)$$
$$= 2n^2 - n$$

$$2\binom{n}{2} + n^2 = 2\frac{n!}{\underbrace{2!}_{2}(n-2)!} + n^2 = \frac{n\cdot(n-1)\cdot\cancel{(n-2)\cdots 1}}{\cancel{(n-2)\cdots 1}} + n^2 = n\cdot(n-1) + n^2$$
$$= n^2 - n + n^2$$
$$= 2n^2 - n$$

√6.4.22 * Combinatorial proof for $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

▷ Hint: commitee of size n from n math and n CS prof's, Such that chairman is from math.

▷ LHS: Consider cases for how many math to include. Let K be # math to include. we have

$k=2$: $\underbrace{\binom{n}{2}}_{\text{choose math}} \cdot \underbrace{\binom{n}{2}}_{\text{CS } \underline{\text{not}} \text{ to choose}} \cdot \underbrace{2}_{\text{ways to choose chairman}}$, $\binom{n}{2} = \binom{n}{n-1}$

Combine all K: $\sum_{k=1}^{n} \binom{n}{k}^2 \cdot k$

▷ RHS: choose math chairman: n
among rest $2n-1$, choose commitee of $n-1$ $\Rightarrow n\cdot\binom{2n-1}{n-1}$

Generalized perm & comb, repetition, indist / dist

√6.5.2 * # strings of 6 letters

▷ we already know: 26 for first choice, 26 for next... $\Rightarrow 26^6$

▷ Can we see it as permutation or combination (with or without repetition?

▷ Thm 6.5.1: r-Perm of n objects $\Rightarrow n^r$

Objs? 26 letters

"boxes"? Slots $\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}$ must be filled with letters, we can use each letter multiple times.

Notice: Order (of slots) matters, $ABCDEF \neq BACDEF$

√6.5.4 * # ways to select 5 unordered elems from set of size 3? with repetition

▷ Look at table 1. we know it is w. rep, but is it perm or comb?

Perm: order matters, Comb: order doesn't matter

▷ It is $\underset{r}{5}$- comb of $\underset{n}{3}$-set $\Rightarrow \binom{n+r-1}{r} = \binom{3+5-1}{5}$

✓ 6.5.6 Bagel Shop: onion, Poppy Seed, Egg, Salty, Pumpernickel, sesame, raisin, plain } unlimited

1 onion, 2 Poppy Seed, 3 Egg, 4 Salty, 5 Pumpernickel, 6 sesame, 7 raisin, 8 plain

\# of ways to choose

a) 6 bagels

- ▷ Perm vs. comb? Does order matter? Bag of bagel, no order
- ▷ Rep vs. No rep? unlimited bagels of each kind, Repetition.
- ▷ $\binom{n+r-1}{r}$, n? r? our set of elements is the 8 kinds, $n=8$
  the amount we draw is 6, $r=6$

▷ $\binom{8+6-1}{6} = 1716$

b) Dozen bagels = 12 bagels ⇒ ▷ n? r? ▷ $n=8$, $r=12 \Rightarrow \binom{8+12-1}{12} = 50388$

c) Two dozen bagels = 24 ⇒ ▷ $n=8$, $r=24 \Rightarrow \binom{8+24-1}{24} = 2629575$

d) 12 (dozen) w. at least one of each kind.

▷ 8 choices already made for us. 4 choices left: $\binom{8+4-1}{4} =$

e) 12 w. at least ($\geq$) 3 egg and no more than ($\leq$) 2 salty.

▷ Again 3 choices made ⇒ 9 left

▷ Do cases for 0 salty, 1 salty, 2 salty; add together
With this, only 7 kinds left

$$\binom{7+9-0-1}{9-0} + \binom{7+9-1-1}{9-1} + \binom{7+9-2-1}{9-2} = 9724$$

6.5.8 pennies, nickels, dimes, quarters, half-dollar →  of size 20

1 pennies, 2 nickels, 3 dimes, 4 quarters, 5 half-dollar

▷ Denom ⇔ Bagel kinds ⇒ $n=8$

▷ Piggy bank size ⇔ Bagel bag ⇒ $r=20$ ⟹ $\binom{8+20-1}{20} = 888030$

✓ 6.5.10 \# of solutions to

$$X_1 + X_2 + X_3 + X_4 = 17, \quad X_i \geq 0, \; X_i \text{ int}$$

▷ See example 5, p. 414: 3 vars ⇒ $n=3$
result 11 ⇒ $r=11$ ⇒ $\binom{3+11-1}{11}$

▷ Now 4 vars ⇒ $n=4$
result 17 ⇒ $r=17$ ⇒ $\binom{4+17-1}{17} = 1140$

√6.5.12 # of ternary strings (0/1/2) of len 10 w. exactly 2×0's, 3×1's, 5×2's

*

$2+3+5=10$

▷ 10 slots _ _ _ _ _ _ _ _ _ _

▷ 10 numbers 0 0 1 1 1 2 2 2 2 2

▷ Find # permutations of ↑, but don't tell diff between 0's, 1's and 2's.

▷ Similar to example 6.5.7, p. 415, with "success"

▷ $\binom{10}{2}$ ways to place 0's, 8 slots left

▷ $\binom{8}{3}$ ways to place 1's, 5 slots left

▷ $\binom{5}{5}$ ways to place 2's, 0 slots left

▷ In total: $\binom{10}{2}\cdot\binom{8}{3}\cdot\binom{5}{5} = \frac{10!}{2!\,\cancel{8!}}\cdot\frac{\cancel{8!}}{3!\,\cancel{5!}}\cdot\frac{\cancel{5!}}{5!\,\cancel{0!}} = \frac{10!}{2!\,3!\,5!} = 2520$

▷ Gives formula for Thm 3 & 4

▷ Corresponds to objects in boxes:

objects: 10 slots

Boxes: $\square$ $\square$ $\square$ (labels)

0's 1's 2's

} A slot is given a label/assignment ⟺ slot put in box

√6.5.14 $x_1 + x_2 + x_3 \leq 11$ (Note similar to example 6.5.5, p. 414)

▷ Use hint: Introduce $x_4 \Rightarrow x_1+x_2+x_3+x_4 = 11$

DM554, x_4 slack variable

▷ Now do same as example 5: $x_1+x_2+x_3 = 11 \Rightarrow \binom{3+11-1}{11}$

▷ For our case: $\binom{4+11-1}{11} = 364$

▷ It corresponds to stars and bars: $\overset{x_1}{xxx} \mid \overset{x_2}{xxx} \mid \overset{x_3}{xxxxx}$

Bars: 3 − 1

Stars: 11

6.5.20 Count numbers $0 < n < 1$ million, where n has exactly 1 digit $= 9$ and $\sum$ digits $= 13$

- ▷ $< 1M \Rightarrow$ 999999, 6 slots
- ▷ Example : 111109
- ▷ Note : As 9 always present, rest must sum to $13 - 9 = 4$
- ▷ How many ways to place the 9? 6
- ▷ Consider remaining 5 slots.

  One could consider cases: $1 \times \boxed{4}$ rest 0, $4 \times \boxed{1}$ rest 0, ...
- ▷ Realize rest can be seen as

$$x_1 + x_2 + x_3 + x_4 + x_5 = 4$$

- ▷ Stars: 4

  Bars: 4 (5-1) $\Rightarrow \binom{5+4-1}{4} = 70$
- ▷ In total: $6 \cdot 70 = 420$