Announcements

First exam project online

Count 10 | DMSS1 | 5 | 07-10-16

Today

Random variables

Expected value, variance

Inequalities (bounds)

Incl-excl (onto func)

7.4.6 E(Sum of 🎲 ×3)? $\boxed{\text{Similar to Example 7.4.1}}$

*

Let D_1, D_2, D_3 be random var, D_i = # eyes on dice i.

So $E(D_1 + D_2 + D_3) = E(D_1) + E(D_2) + E(D_3)$

Linearity of expectation, Thm 7.4.3, p. 466

$E(D_i)$?

$E(X) = \sum_{r \in X(S)} p(X=r) \cdot r = \sum_{i=0}^{K} p(X=i) \cdot i$ ← if X can have values $0, 1, 2, \dots, K$

$E(D_i) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \dots + \frac{1}{6} \cdot 6 = \frac{1}{6}(\underbrace{1+2+\dots+6}_{21}) = \frac{7}{2} \approx 3{,}5$

$E(D_1 + D_2 + D_3) = 3 \cdot \frac{7}{2} = \frac{21}{2} \approx 10{,}5$

7.4.10 Roll 🎲 until 6 comes up $\boxed{\text{Similar to ex 10}}$

a) p(doing n rolls)? $\frac{5}{6}$ for first $n-1$, then $\frac{1}{6}$ for last $\Rightarrow \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$

b) E(# of rolls)? Geometric distribution, def 7.4.2

$\Rightarrow \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$

7.4.22 $V(\#$ times 6 appears on $\times 10)$?

▷ Variance: How spread out / far away from $E(\cdots)$ we will expect occurrences to be

▷ $V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s) \overset{\text{Thm 7.4.6}}{=} E(X^2) - E(X)^2$

▷ Let $X_i = \begin{cases} 1 & \text{if } _i \text{ rolled } 6 \\ 0 & \text{otherwise} \end{cases}$, $X = X_1 + X_2 + \cdots + X_{10}$

indicator variable

Now the question is $V(X)$??

▷ By Bienaymé's formula as X_i's are indep.

$$V(X) = V(X_1 + X_2 + \cdots + X_{10}) = V(X_1) + V(X_2) + \cdots + V(X_{10})$$

▷ $V(X_i)$?

$$V(X_i) = E(X_i^2) - E(X_i)^2$$

as outcomes are just $0, 1 \Rightarrow 0^2 = 0, 1^2 = 1$

$$= E(X_i) - E(X_i)^2$$

$E(X_i) = p(X_i = 1)$

$$= \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{5}{36}$$

▷ $V(X) = \underbrace{V(X_1) + \cdots + V(X_{10})}_{10} = 10 \cdot \frac{5}{36} = \boxed{\frac{50}{36}}$

▷ <u>Alternative</u>: we can see this as Bernoulli trials

Success = Roll 6, $p = \frac{1}{6}$, $n = 10$ trials

Look at Jørgens notes on indicator vars, <u>week 4 note 4</u>

$V(X) = n\,p \cdot (1 - p)$

7.4.28 Soda pops bottling plant

X = # of cans filled per day

we know

$E(X) = 10\,000$

$V(X) = 1\,000$

a) Upper bound on $p(X \geq 11000)$

Markov Ineq: $p(X \geq a) \leq \frac{E(X)}{a}$

$p(X \geq 11000) \leq \frac{10\,000}{11\,000} = \frac{10}{11}$

b) Lower bound on $p(9000 \leq X \leq 11000) = p(|X - 10000| < 1000)$

we look at complement $p(|X - 10000| \geq 1000)$

Chebyshev Ineq: $p(|X - E(X)| \geq r) \leq \frac{V(X)}{r^2}$

$p(|X - 10000| \geq 1000) \leq \frac{1000}{1000^2} = \frac{1}{1000}$ ← upperbound on complement

Result: $1 - \frac{1}{1000}$ ← lower bound

7.4.33 Hat check problem aka. fixed elems in permutation of n elem

Variance of # of fixed elems?

let $X_i = \begin{cases} 1 & \text{if elem } i \text{ same before and after perm} \\ 0 & \text{otherwise} \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$

The question is now: $V(X)$? Compute $E(X^2) - E(X)^2$

▷ $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$, $E(X_i) = p(X_i = 1) = \frac{1}{n}$

$= n \cdot \frac{1}{n} = 1$

$E(X)^2 = 1^2 = 1$

▷ For $E(X^2)$, we multiply out $X^2 = (X_1 + X_2 + \cdots + X_n)^2$

Continued

7.4. 33 continued

▷ For $E(X^2)$, we multiply out $X^2 = (X_1 + X_2 + \cdots + X_n)^2$

$$= (X_1 + X_2 + \cdots + X_n)(X_1 + X_2 + \cdots + X_n)$$

$$= X_1(X_1 + X_2 + \cdots + X_n) + X_2(X_1 + \cdots + X_n) + \cdots + X_n(X_1 + \cdots + X_n)$$

$$= X_1^2 + X_1X_2 + \cdots + X_1X_n + X_2^2 + X_2X_1 + \cdots X_2X_n + \cdots$$

$$= \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_iX_j$$ ← all pairs of i, j where $i \neq j$

Note: $X_i^2 = X_i$ when X_i is indicator variable

▷ $E(\sum X_i^2) = \sum E(X_i^2) = \sum E(X_i) = \sum \frac{1}{n} = n \cdot \frac{1}{n} = 1$

▷ $E(\sum X_iX_j)$? what are possibilities for X_iX_j?

		X_i = 0	X_i = 1
X_j	0	0	0
	1	0	1

X_iX_j only 1 when both elem i and j are same place before and after permutation (i.e. gets their hat back)

$P(X_iX_j = 1)$? Must be $\frac{\#\text{ of perms of } n-2 \text{ obj}}{\#\text{ of perms of } n \text{ obj}} = \frac{(n-2)!}{n!}$

$$= \frac{\cancel{(n-2)!}}{n \cdot (n-1) \cdot \cancel{(n-2)!}} = \frac{1}{n \cdot (n-1)}$$

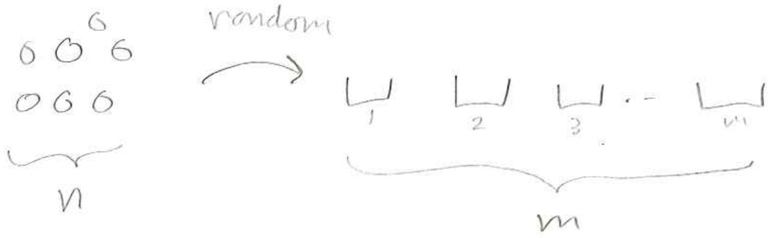
\# of pairs

$E(X_iX_j) = \frac{1}{n \cdot (n-1)}$, $E(\sum X_iX_j) = n \cdot (n-1) \cdot \frac{1}{n \cdot (n-1)} = 1$

$E(\sum X_i^2 + \sum X_iX_j) = 1 + 1 = 2$

$V(X) = E(X^2) - E(X)^2 = 2 - 1 = 1$

7.4.38



Exp balls in box 1?

Focus on box 1

let

$X_i = \begin{cases} 1 & \text{if obj } i \to \text{box 1} \\ 0 & \text{otherwise} \end{cases}$; $E(X_i) = P(X_i = 1) = \frac{1}{m}$

$$X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{m} = \frac{1}{m} \sum_{i=1}^{n} 1 = \frac{1}{m} \cdot n = \frac{n}{m}$$

Alternative: Apply Exp Bernoulli (Thm 7.4.2)

Success = Obj in box 1 , $p = \frac{1}{m}$, n trials

$\Rightarrow$ $E(\#\text{ success, } n \text{ trials}) = n \cdot p = n \cdot \frac{1}{m}$

SE 7.14

$P(\text{random bit str of len 10 is palindrome})$?

$\#$ palindromes $= 2^5$

$\#$ bit str $= 2^{10}$ $\Rightarrow$ prob $= \frac{2^5}{2^{10}} = \frac{1}{2^5} = \frac{1}{32}$

SE 7.24

Use Chebyshev to show $P(\underbrace{> 10 \text{ ppl get hat back}}_{X}) \le \frac{1}{100}$

[use example 7.4.6 and exercise 7.4.33]

We know

$E(X) = 1$

$V(X) = 1$

i.e. $X \geq 10$

$$P(|X - 1| \geq 10) \leq \frac{V(X)}{10^2} = \frac{1}{100}$$

SE 7.26 $m \in \mathbb{N}$, $p(m\text{'th success in } m+n \text{ trials})$

$$= \binom{n+m-1}{n} q^n p^m$$

▷ Insert in Bernoulli, but with one less trial:

$\binom{n}{r} p^r q^{n-r}$, where $n = \#\text{ trials} = n+m-1$

$r = m-1$

▷ $$\binom{n+m-1}{m-1} p^{m-1} q^{\cancel{n+m-1-(m-1)}} = \binom{n+m-1}{n} p^{m-1} q^n$$

because $\binom{n}{r} = \binom{n}{n-r}$, this becomes

▷ Now multiply by p to take success of the m'th trial that we had left out before

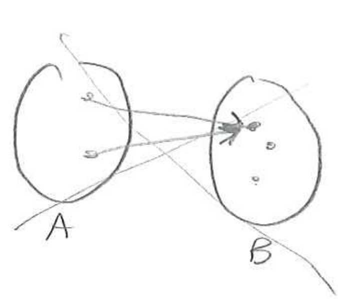
▷ $$\binom{n+m-1}{n} p^{m-1} q^n \cdot p = \binom{n+m-1}{n} p^m \cdot q^n$$

January 11.1 Show $n! = n^n + \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (n-i)^n$

Hint: Count # of 1-1 (one to one) and onto functions.

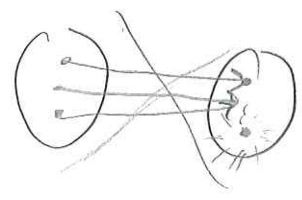
▷ One-to-one (Injective): we cannot have two mapping to same

In other words, each elem in B is hit at most once



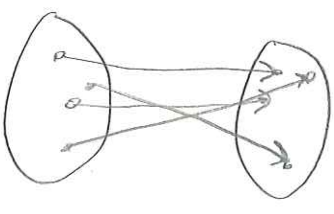
▷ Onto (Surjective): we have to hit all elems in B

In other words, each elem in B is hit at least once



▷ one-to-one + onto (bijective):

Acts as a permutation.

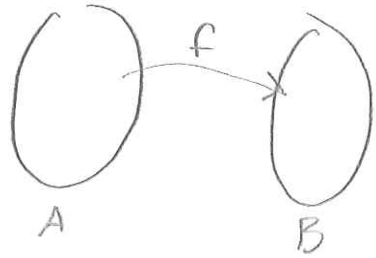


Continued

Jan 11.1 Continued $\quad n! = n^n + \sum_{i=1}^{n-1}(-1)^i \binom{n}{i}(n-i)^n$

Δ Now we can start to argue

$\triangleright$ How many bijective functions?

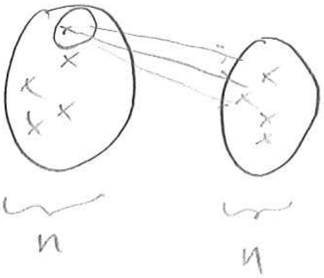


Acts as permutation so $n!$

This is left-hand side.

$\triangleright$ How many functions in total? $\quad |A| = |B| = n$

(no injective or surjective requirement)



For first elem in A, it has n choices in B.

For second elem, it has n choices etc.

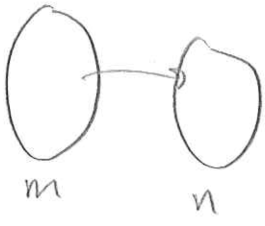
$\underbrace{n \cdot n \cdots n}_{n} = n^n$

This gives part of right-hand side.

$\triangleright$ How many are <u>not</u> bijective?

Look at p. 544, Section 8.6 "# of onto functions"

Thm 8.6.1



$n^m - \binom{n}{1}\cdot(n-1)^m + \binom{n}{2}\cdot(n-2)^m - \dots + (-1)^{n-1}\binom{n}{n-1} 1^m$

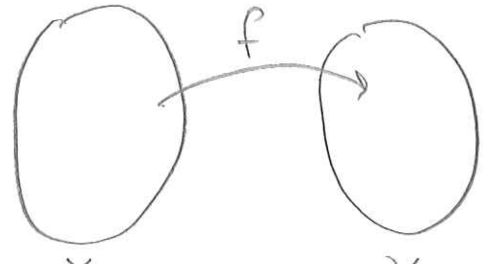
In our case $m = n$ $\quad = \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m$

Also, first term is total # of functions, so rest must be <u>non</u>-bijective functions.

So $\sum_{i=1}(-1)^i\binom{n}{i}(n-i)^n$ is # of <u>non</u>-bijective func

which gives the second part of right-side $\quad$ with minus in front

Jan 11.4 Again # of onto functions



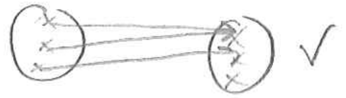
X $\quad$ Y

$|X| = m$ $\quad$ $|Y| = n$

p. 544

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m$$
$$+ \cdots + (-1)^{n-1}\binom{n}{n-1} 1^m$$
$$= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m$$

a) # of ways to



So ≥ 1 box is empty?

This is the same as counting all <u>non</u>-onto functions

$\Rightarrow$ $m = K$

$n = 5$, $\quad \binom{5}{1} \cdot (5-1)^K - \binom{5}{2}(5-2)^K + \binom{5}{3}(5-3)^K$

$- \binom{5}{4} \cdot (5-4)^K = 5 \cdot 4^K - 10 \cdot 3^K + 10 \cdot 2^K$
$- 5 \cdot 1^K$

b) # of ways to

So exactly one box is empty?

Just consider only 4 boxes with no box empty. and multiply by the choice of the last box: $\binom{5}{1} = 5$. [Use all terms of formula]

$m = K$

$n = 4$, $\quad 4^K - \binom{4}{1}(4-1)^K + \binom{4}{2}(4-2)^K - \binom{4}{3}(4-3)^K$

$= 4^K - 4 \cdot 3^K + 6 \cdot 2^K - 4 \cdot 1^K$

Result: $\binom{5}{1} \cdot (4^K - 4 \cdot 3^K + 6 \cdot 2^K - 4 \cdot 1^K)$

Continued

Jan 11.4 continued

c) a) was ≥ 1 box empty
i.e. exactly 1 or exactly 2
or exactly 3 or exactly 4

b) was exactly 1 box empty

So we just take a) − b)
to get ≥ 2 boxes empty

# of ways with
X boxes empty

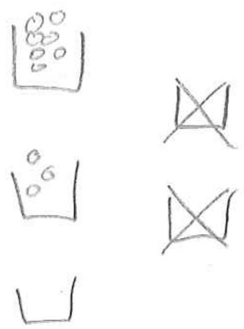
Thm 8.6.1	0	1	2	3	4	5
a)	0	1	2	3	4	5
b)	0	1	2	3	4	5
c)	0	1	2	3	4	5

↑ not possible

d) Why is answer to c) <u>not</u> $\binom{5}{2}3^k$?

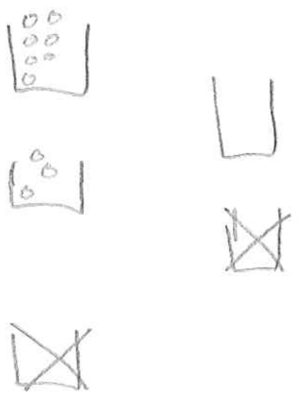
Overcounting!

Consider one particular distribution



: box chosen to be empty

But this is the same as



We do not distinguish between if we chose
the box to be empty or no obj was put in it,
so we count it twice.