Count
14

| DMSS 1 | 6 | 12-10-16 |
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## Announcements

Last exercise session before assignment hand-in.

| | |
|---|---|
| Linearity of exp | p. 466 |
| Bernoulli, exp | p. 465 |
| Exp value, formula | p. 464 |
| Prob, conditional | p. 442 |
| Incl-Excl | p. 541 |
| # of onto func | p. 543 |
| Ind. var. | weekly note 4 |

## Today

Only old exam problems,
Focus on relevant parts for assignment
Incl-Excl

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Jan 10. 1 Let $n \in \{1, 2, \dots, 999\}$

Find # of diff n's with no common digits.

Common digit: 1 7 1

Cases: ← And then we sum all cases.

$1 \le n \le 9$: No numbers w. common digits $\Rightarrow$ 9

$10 \le n \le 99$: 11, 22, ... 99 have common digits

There are 90 in total and 9 of ↑ $\Rightarrow 90 - 9 = 81$

$100 \le n \le 999$: $\overset{1}{\square}\overset{2}{\square}\overset{3}{\square}$, 9 choices for $\overset{1}{\square}$ (cannot be 0)

then 9 choices for $\overset{2}{\square}$ (can be 0, but not same as $\overset{1}{\square}$)

then 8 choices for $\overset{3}{\square}$ (cannot be same as $\overset{1}{\square}$ and $\overset{2}{\square}$)

$9 \cdot 9 \cdot 8 = 648$

In total: $9 + 81 + 648 = 738$

Jan 10.4 We have $\boxed{1|2|3|4|5}$, want to perform permutation $\pi$

where $\pi(i+1)-\pi(i) \neq 2$ for $i = 1, 2, 3, 4$

Example $\boxed{1|2|3|4|5} \longrightarrow \boxed{3|1|4|2|5}$

$1-3=-2$ $\quad 4-1=3$ $\quad 2-4=-2$ $\quad 5-2=-3$

We have to use inclusion-exclusion!
Specifically the counting technique on page 541 w. $N(P_1'P_2' \dots P_k')$
One might be tempted to define $P_i : \pi(i+1)-\pi(i) = 2$ for $i = 1, \dots, 4$
and then the result is $N(P_1'P_2'P_3'P_4')$ ← this gives 16 terms to figure out!
We can be smarter!
What pairs are actually allowed by $\pi(i+1)-\pi(i)=2$?
$\boxed{1|3}$ $\boxed{2|4}$ $\boxed{3|5}$

So let

$P_1$ : permutations where $\boxed{1|3}$ is a substring → ex $\boxed{5|2|1|3|4}$

$P_2$ : permutations where $\boxed{2|4}$ is a substring → ex $\boxed{5|3|1|2|4}$

$P_3$ : permutations where $\boxed{3|5}$ is a substring → ex $\boxed{3|5|2|1|4}$

We must now find $N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$

$N = 5! = 120$ (total # of perm)

$N(P_i) = 4 \cdot 3! = 24$

(4: choice of where to put $\boxed{1|3}$, $\boxed{2|4}$ or $\boxed{3|5}$; $3!$: perm of rest)
$\;\;i=1 \quad i=2 \quad i=3$ ; Key idea: See $\boxed{1|3}$ etc as <u>1 element</u>.

$N(P_1P_2) = 3! = 6$

↖ perm of $\boxed{1|3}$, $\boxed{2|4}$ and $\boxed{5}$

$N(P_2P_3) = 3! = 6$

↖ perm of $\boxed{2|4}$, $\boxed{3|5}$ and $\boxed{1}$

$N(P_1P_3) = 3 \cdot 2! = 6$

(3: choice for $\boxed{1|3|5}$; $2!$: perm of rest)

$N(P_1P_2P_3) = 2! = 2$

↖ perm of $\boxed{1|3|5}$ and $\boxed{2|4}$

In total

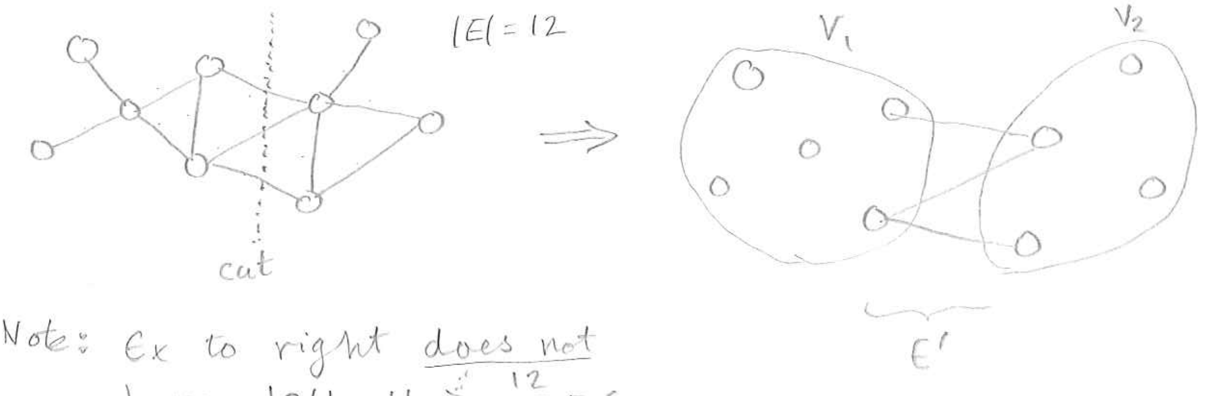
$$N(P_1'P_2'P_3') = 120 - 3 \cdot 24 + 3 \cdot 6 - 2 = 64$$

Jan 11. 5 2-partition of graph

$V_1 \cap V_2 = \emptyset$
$V_1 \cup V_2 = V$

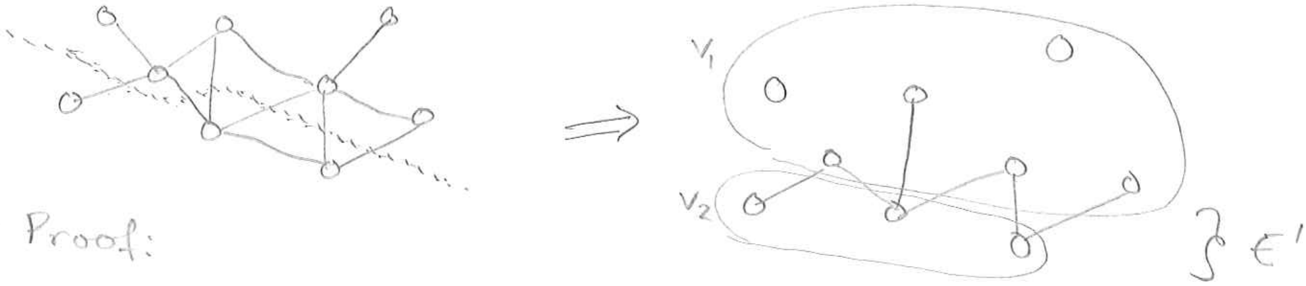
For $G=(V,E)$, show $\exists$ subgraph $H=(V,E')$, $E' \subseteq E$ where $H$ is 2-partition and $|E'| \geq \frac{|E|}{2}$

▷ Example of 2 partition



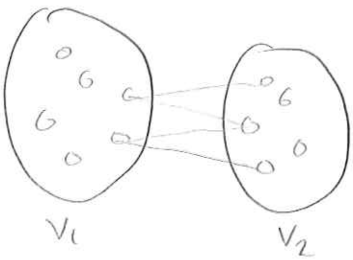

Note: Ex to right does not have $|E'| = 4 \ngeq \frac{12}{2} = 6$

▷ Example where $|E'| \geq \frac{|E|}{2}$




▷ Proof:

Consider random 2-partition $H$ of $G$. ← For instance, let an alg go through all $v \in V$ and at random put $v$ in $V_1$ or $V_2$ with $\frac{1}{2}$ prob.




- What is prob that some $e \in E$ is in $E'$ as well? In other words, $e$ has an endpoint in $V_1$ and another in $V_2$. Consider $e=(u,v) \in E$ ; $u,v \in V$. WLOG $u$ was placed in $V_1$. what is prob of $v$ being placed in $V_2$? $\frac{1}{2}$

So prob of $e$ being in $E'$ is $\frac{1}{2}$

Continued

Jan 11. 5 continued

- Now let $X_e = \begin{cases} 1 & \text{if } e \in E' \\ 0 & \text{otherwise} \end{cases}$ (ind. var.)

$E(X_e) = p(X_e = 1) = \frac{1}{2}$

- Let $X = |E'|$, but $X$ can also be expressed as $X = \sum_{e \in E} X_e$

why? Each $X_e$ contribute with 1 if $e$ in $E'$ effectively counting # of elem in $E'$.

- We compute $E(X) = E\left(\sum_{e \in E} X_e\right) = \sum_{e \in E} E(X_e) = \sum_{e \in E} \frac{1}{2} = \frac{|E|}{2}$

- We now suppose towards contradiction that

$|E'| < \frac{|E|}{2}$ for all subgraphs $H$ of $G$

$\left(\text{ie } |E'| \leq \frac{|E|}{2} - 1\right)$

Obs: $p\left(X \geq \frac{|E|}{2}\right) = 0$ in that case

We now consider $E(X)$ using normal formula for exp value:

$$E(X) = \sum_{j=0}^{|E|} j \cdot p(X=j) = \sum_{j=0}^{\frac{|E|}{2}-1} j \cdot p(X=j) + \sum_{j=\frac{|E|}{2}}^{|E|} j\, \underbrace{p(X=j)}_{\text{for } j \geq \frac{|E|}{2} \text{ this is } 0}$$

j is outcomes of X

Split sum

$$= \sum_{j=0}^{\frac{|E|}{2}-1} j\, p(X=j) + 0 \leq \sum_{j=0}^{\frac{|E|}{2}-1} \underbrace{\left(\frac{|E|}{2} - 1\right)}_{\text{replace with largest value of } j} p(X=j)$$

$$= \left(\frac{|E|}{2} - 1\right) \cdot \underbrace{\sum p(X=j)}_{\leq 1, \text{ sum of probs must be } \leq 1}$$

$$= \frac{|E|}{2} - 1$$

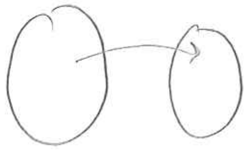
So $E(x) \leq \frac{|E|}{2} - 1$, but we showed that $E(X) = \frac{E}{2}$ $\lightning$

- Thus there exists some subgraph w. $|E'| \geq \frac{|E|}{2}$

∎

Jan 11.6 We consider random func/map $f: S \to T$

(a)−(g)



S T

$|S|=|T|=n \geq 2$

Generation (Q):

for $s \in S$:

pick $t \in T$ at random $\left(\frac{1}{n}\right)$

set $f(s)=t$

Think about this as hashing where $m=n$ and we remember choice of box for each element.

Now let $f^{-1}(t) = \{ s \in S \mid f(s)=t \}$

Obs: $0 \leq |f^{-1}(t)| \leq n$

and $\sum_{t \in T} |f^{-1}(t)| = n$

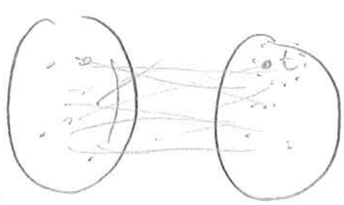
a) Prob that f is 1-1 and onto? $\frac{n!}{n^n}$ ← # of 1-1 + onto func ← Total # of func

b) $E$(# of s's that map to some t)? ← $|f^{-1}(t)|$

Fix t.

Focus on t

Let $X_{t,s} = \begin{cases} 1 & \text{if } s \text{ map to } t \\ 0 \end{cases}$



How many hit t?

$E(X_{t,s}) = P(X_{t,s}=1) = \frac{1}{n}$

Let $X = |f^{-1}(t)|$, but X can also be expressed as

$X = \sum_{s \in S} X_{t,s}$. Why? Each $X_{t,s}$ contribute with 1 if s maps to t and is thus counted in $|f^{-1}(t)|$

$E(X) = E\left(\sum X_{t,s}\right) = \sum E(X_{t,s}) = \sum_{s \in S} \frac{1}{n} = \frac{1}{n} \sum_{s \in S} 1 = \frac{1}{n} \cdot |S| = \frac{1}{n} \cdot n = 1$

c) Let event $U_t$ be that $|f^{-1}(t)|=1$, i.e. all other elem in S don't hit t, and then s hit t.

$P(U_t)$? Prob of not hit: $\frac{n-1}{n}$

n ways to choose $s \in S$ to hit t, with prob $\frac{1}{n}$, then $\left(\frac{n-1}{n}\right)^{n-1}$

to not hit t more $\Rightarrow n \cdot \frac{1}{n} \cdot \left(\frac{n-1}{n}\right)^{n-1} = \left(\frac{n-1}{n}\right)^{n-1}$ continued

Jan 11. 6. continued

d) $p(U_{t'} | U_t)$? for $t' \neq t$ $\quad p(U_{t'} | U_t) = \dfrac{p(U_{t'} \cap U_t)}{p(U_t)}$ ← we have this from c)

$p(U_{t'} \cap U_t)$?

Here we have to pick $s$ and $s'$ to hit $t$ and $t'$ ← $n \cdot (n-1)$ ways

Prob of this is $\underbrace{\frac{1}{n}}_{\text{hit } t} \cdot \underbrace{\frac{1}{n}}_{\text{hit } t'} \cdot \underbrace{\left(\frac{n-2}{n}\right)^{n-2}}_{\text{don't hit } t \text{ and } t' \text{ again}}$

$$p(U_{t'} \cap U_t) = \cancel{n} \cdot (n-1) \cdot \frac{1}{\cancel{n}} \cdot \frac{1}{n} \cdot \left(\frac{n-2}{n}\right)^{n-2}$$

$$p(U_{t'} | U_t) = \frac{(n-1)\frac{1}{n} \cdot \left(\frac{n-2}{n}\right)^{n-2}}{\left(\frac{n-1}{n}\right)^{n-1}} = \cdots = \left(\frac{n-2}{n-1}\right)^{n-2}$$

e) Are $U_t$ and $U_{t'}$ indep? Intuition: no, if we know some $t$ was hit, then we have fewer $s$'s to worry about hitting $t'$!

compare $p(U_{t'} | U_t)$ and $p(U_{t'})$

$$\left(\frac{n-2}{n-1}\right)^{n-2} \neq \left(\frac{n-1}{n}\right)^{n-1}$$, so no, not indep.

f) $E(\#$ of $t$'s where $U_t$ happen$)$?

Let $A_t = \begin{cases} 1 & \text{if } U_t \\ 0 \end{cases}$, $\quad E(A_t) = p(U_t) = \left(\frac{n-1}{n}\right)^{n-1}$

$A = \sum_{t \in T} A_t$, $\quad E(A) = E(\sum A_t) = \sum E(A_t) = \sum_{t \in T} \left(\frac{n-1}{n}\right)^{n-1}$

$= |T| \cdot \left(\frac{n-1}{n}\right)^{n-1} = n \cdot \left(\frac{n-1}{n}\right)^{n-1}$

g) We now repeat Q $r$ times. $E(\#$ of times $U_t$ occur$)$? $X_t$

Apply Bernoulli: Each Q is indep, so $r$ indep trials

Prob of succes $= p(U_t)$

$\Rightarrow r \cdot p(U_t) = r \cdot \left(\frac{n-1}{n}\right)^{n-1}$

Jan 15. 3 Ternary Strings

Example 001221

a) # of ternary str of len n?

3 choices for 1st
3 choices for 2nd
$\vdots$
} product rule

$\Rightarrow \underbrace{3 \cdot 3 \cdots 3}_{n} = 3^n$ (just like $2^n$ for binary str)

b) # of ternary str of len n w. $\geq 1 \times \boxed{0}$, $\geq 1 \times \boxed{1}$, $\geq 1 \times \boxed{2}$

We use inclusion-exclusion:

Let

$P_0$ : $< 1 \times \boxed{0}$ ie. exactly $0 \times \boxed{0}$ ie no $\boxed{0}$'s
$P_1$ : $< 1 \times \boxed{1}$ ie. exactly $0 \times \boxed{1}$ ie no $\boxed{1}$'s
$P_2$ : $< 1 \times \boxed{2}$ ie. exactly $0 \times \boxed{2}$ ie no $\boxed{2}$'s

Our answer is $N(P_0' P_1' P_2') = N - N(P_0) - N(P_1) - N(P_2)$ a)
$+ N(P_0 P_1) + N(P_0 P_2) + N(P_1 P_2)$
$- N(P_0 P_1 P_2)$

$N(P_0) =$ "str of only $\boxed{1}$'s and $\boxed{2}$'s" $= 2^n$ (just like bit str)
$N(P_1) =$ (same idea) $= N(P_2) = 2^n$

$N(P_0 P_1) =$ "str of only $\boxed{2}$'s" $= 1^n = 1$
$N(P_0 P_2) = N(P_1 P_2) = 1$

$N(P_0 P_1 P_2) =$ "no digits" $= 0$

$N(P_0' P_1' P_2') = 3^n - 3 \cdot 2^n + 3 \cdot 1 - 0 = 3^n - 3 \cdot 2^n + 3$

Jan 15. 4

a) $m=10 \rightarrow n=4$, so no box is empty

think Stirling or # of onto func

Use formula for # of onto functions

$$\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m = \sum_{i=0}^{3}(-1)^i\binom{4}{i}(4-i)^{10}$$

$$= 4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}\cdot 2^{10} - \binom{4}{3}\cdot 1^{10} = 4^{10} - 4\cdot 3^{10} + 6\cdot 2^{10} - 4$$

b) $m=10 \rightarrow n=4$, so exactly one box is empty

Simply consider if $n=3$ with no box empty and multiply with # of ways to choose the empty box among the 4

$$\binom{4}{1}\left(3^{10} - \binom{3}{1}2^{10} + \binom{3}{2}1^{10}\right)$$

c) 20 dist obj, 10 red, 10 blue, randomly into 4 dist boxes.

P(all boxes contain $\geq 1\times$ O and $\geq 1\times$ ●)?

Treat red and blue separately.

Prob of all boxes containing $\geq 1 \times$ O is # of ways to distribute objs, so no box is empty (i.e. a)), divided by total # of ways to distribute 10 objs in 4 boxes, namely $4^{10}$.

Do same for blue. Prob is product of the two:

$$\frac{4^{10} - 4\cdot 3^{10} + 6\cdot 2^{10} - 4}{4^{10}} \cdot \frac{4^{10} - 4\cdot 3^{10} + 6\cdot 2^{10} - 4}{4^{10}} = \frac{(4^{10} - 4\cdot 3^{10} + 6\cdot 2 - 4)^2}{4^{20}}$$

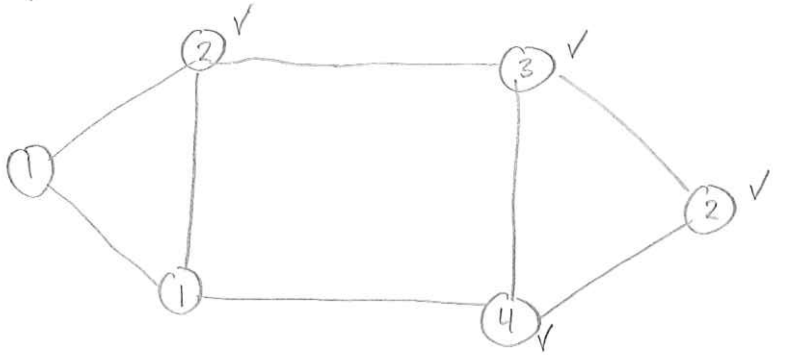
Jan 14. 3 Graph coloring, $\leq 4$ colors

We consider only graphs of degree $\leq 3$
i.e. all nodes have $\leq 3$ neighbors.

Def: Good node: If it has a diff color than all its neighbors.

Colors: 1, 2, 3, 4 (written inside node)

▷ Example



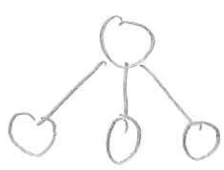

NP-hard problem.

✓: good

▷ Give random alg
which achieves exp # of good $= \frac{27}{64} n$ ✓ $|V|$

Alg:
for $v \in V$:
pick $c \in \{1, 2, 3, 4\}$ (prob $\frac{1}{4}$ for each) and assign color $c$ to $v$.

▷ Consider a node and its neighbors



What is prob that neighbors will get same color as node? $\frac{1}{4}$
So, prob of being good with only one neighbor is $\frac{3}{4}$.
For 3 neighbors prob of good is $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$

▷ Let $x_i = \begin{cases} 1 & \text{if i'th node is good} \\ 0 \end{cases}$

let $x = \sum_{i=1}^{n} x_i$, $E(x_i) = P(x_i = 1) = \frac{27}{64}$

$$E(x) = \sum E(x_i) = \sum_{i=1}^{n} \frac{27}{64} = \frac{27}{64} n$$